Charged Particle Optics. Matrix Representation of the Accelerator Elements

David Robin

Outline

- What are the Optics?
  - Magnet Definitions
  - Magnet Functions
- Particle motion in accelerator
  - Coordinate system
  - Beam guidance
    - Dipoles
  - Beam focusing
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- Hill's equations and Transport Matrices
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  - Quadrupoles
  - Dipoles
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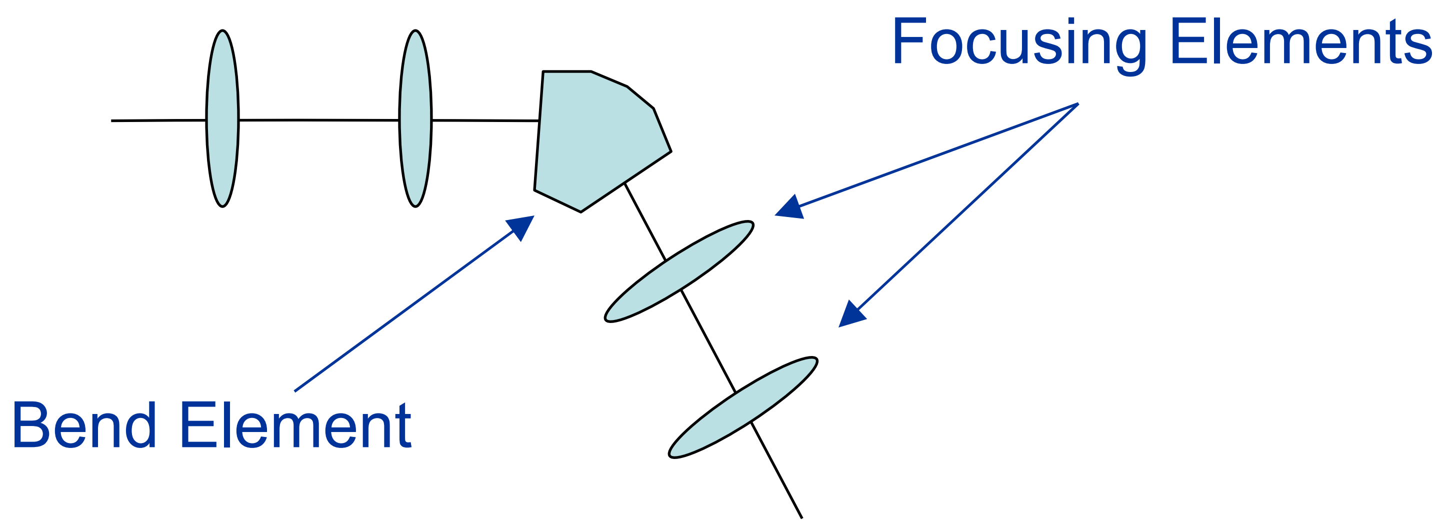
  - FODO

What are the Optics?

- The Optics are the distribution of elements (typically magnetic or electrostatic) that guide and focus the beam - sometimes referred to as the lattice.



Optics or Lattice Design Determines the Beam Properties

Choice of the design depends upon the goal of the accelerator

- Small spot size
- High brightness
- Small divergence
- Obey certain physical constraints (building or tunnel)
- …

Equations of Motion

The motion of each charged particle is determined by the electric and magnetic forces that it encounters as it orbits the ring:

- **Lorentz Force**

$$F = ma = e(E + v \times B),$$

m is the relativistic mass of the particle,

e is the charge of the particle,

v is the velocity of the particle,

a is the acceleration of the particle,

E is the electric field and,

B is the magnetic field.

Two Problems (Inverse Problems)

1. Given an existing lattice, determine the properties of the beam.

2. For a desired set of beam properties, determine the design of the lattice.

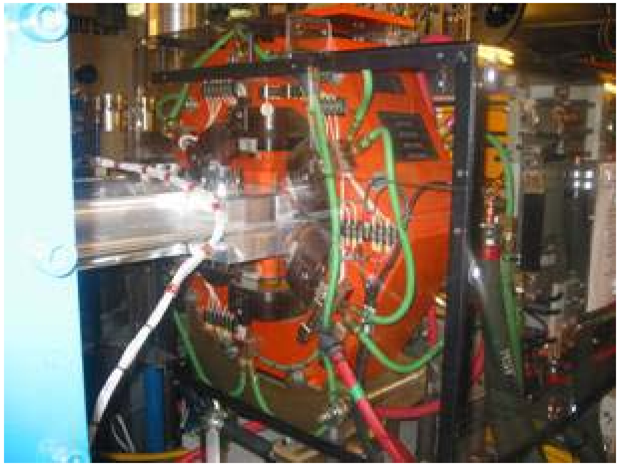
The first problem is in principle straight-forward to solve.

The second problem is not straight-forward – *a bit of an art.*

Magnets to Guide and Focus the Beam

Quadrupoles

Dipoles

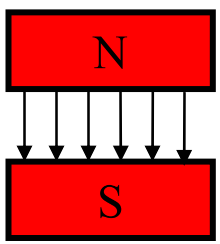


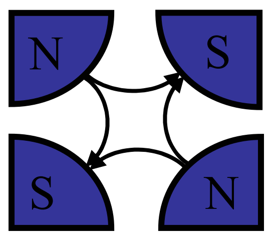
Sextupoles

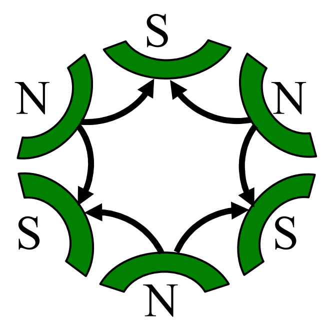
Magnet Definitions

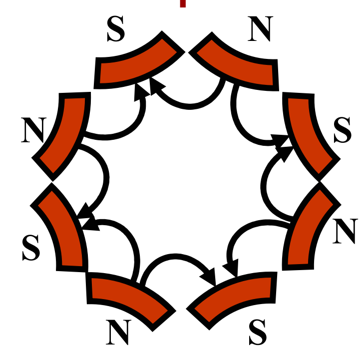
- **2n-pole**:

dipole quadrupole sextupole octupole …









n: 1 2 3 4 …

- Normal: gap appears at the horizontal plane
- Skew: rotate around beam axis by π**/2n** angle
- Symmetry: rotating around beam axis by π**/n** angle, the field is reversed (polarity flipped)

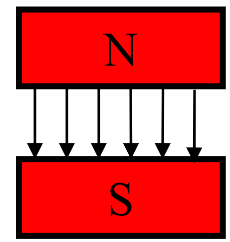
Typical Magnet Types

There are several magnet types that are used in storage rings:

Dipoles → used for guiding

$B_x = 0$

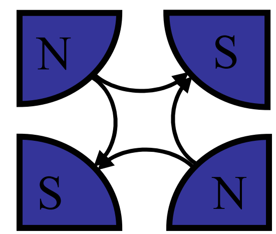
$B_y = B_o$



Quadrupoles → used for focussing

$B_x = Ky$

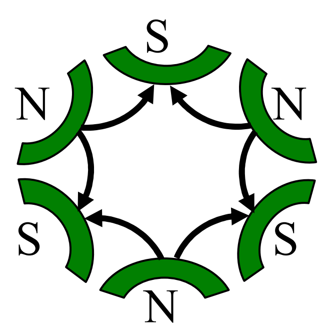
$B_y = Kx$



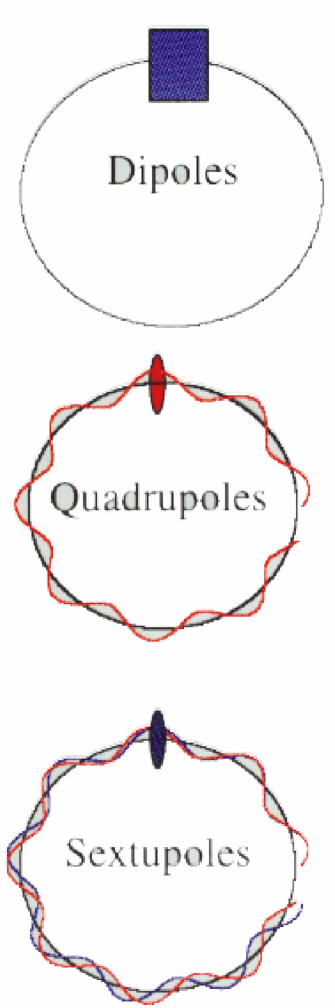
Sextupoles → used for chromatic correction

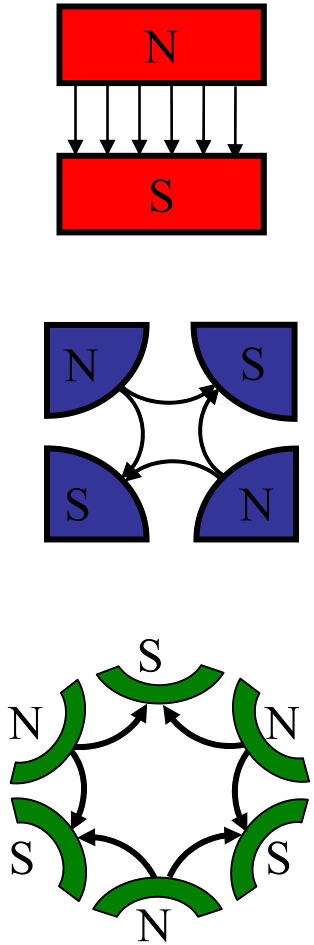
$B_x = 2Sxy$

$B_y = S(x^2 – y^2)$



Functions of the Magnetic Elements





Dependent Variable

In the Lorentz Force Equation as written below, the dependent variable is time, *t*

- **Lorentz Force**

$$F = ma = e(E + v \times B),$$

m **is the relativistic mass of the particle,**

e **is the charge of the particle,**

v **is the velocity of the particle,**

a **is the acceleration of the particle,**

E **is the electric field and,**

B **is the magnetic field.**

Dipoles

- Consider a storage ring for particles with energy *E* with *N* dipoles of length *l*
- The bending angle is

$$\theta = \frac{2\pi}{N}$$

- The bending radius is

$$\rho = \frac{l}{\theta}$$

- The integrated dipole strength will be $$Bl = \frac{2\pi}{N}\frac{\beta E}{q}$$
- By fixing the dipole field, the dipole length is imposed and vice versa
- The highest the field, shorter or smaller number of dipoles can be used
- Ring circumference (cost) is influenced by the field choice

Focusing Elements

- Magnetic element that deflects the beam by an angle proportional to the distance from its centre (equivalent to ray **optics**) provides focusing.
- For a focal length f the deflection angle $\alpha = -\frac{y}{f}$
- A magnetic element with length **l** and with a gradient **g** has a $B_x = gy$ so that the deflection angle is

$$\alpha = -\frac{l}{f} = -\frac{q}{\beta E} B_x l = -\frac{q}{\beta E} gyl$$

- The normalised focusing strength

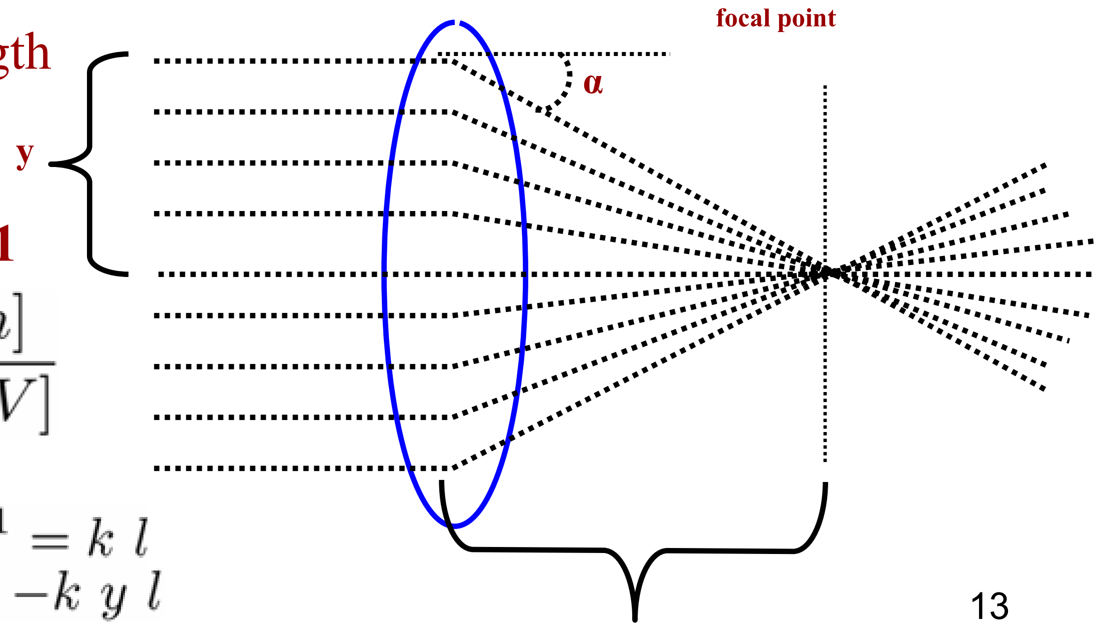
$$k = \frac{qg}{\beta E}$$

- In more practical units, for **Z=1**

$$k[m^{-2}] = 0.2998 \frac{g[T/m]}{\beta E[GeV]}$$

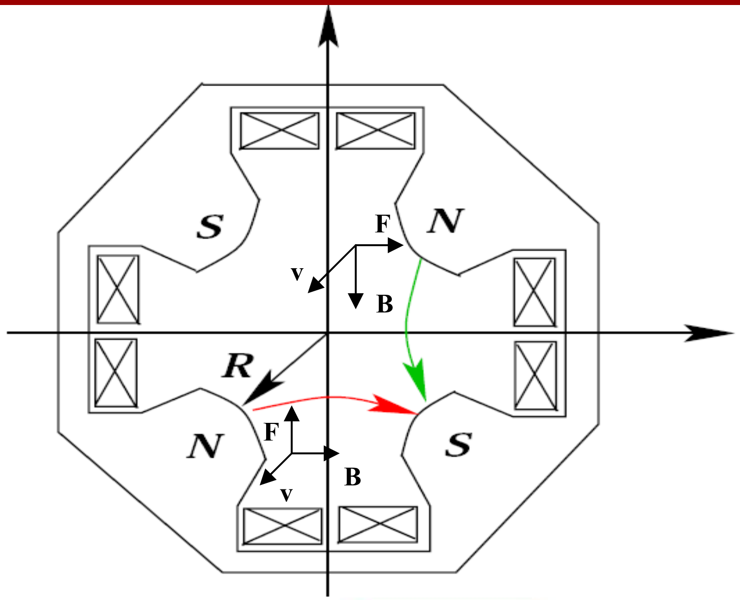
- The focal length becomes $f^{-1} = k\ l$

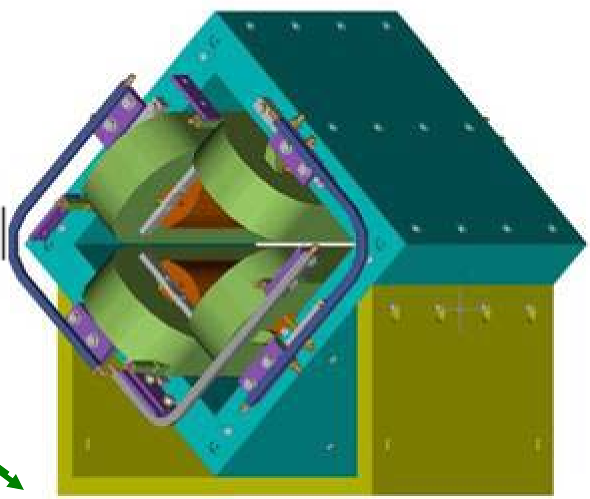
and the deflection angle is $\alpha = -k\ y\ l$

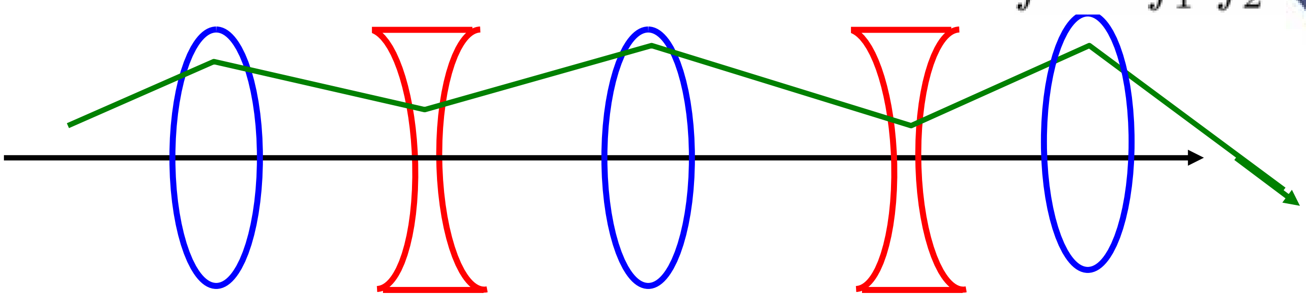


Quadrupoles

- Quadrupoles are focusing in one plane and defocusing in the other
- The field is $(B_x, B_y) = g(y, x)$
- The resulting force $(F_x, F_y) = k(y, -x)$
- Need to alternate focusing and defocusing in order to control the beam, i.e. **alternating gradient focusing**
- From optics we know that a combination of two lenses with focal lengths **f1** and **f2** separated by a distance **d** $$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1\ f_2}$$
- If $f_1 = -f_2$, there is a net focusing effect, i.e. $\frac{1}{f} = |\frac{d}{f_1\ f_2}|$



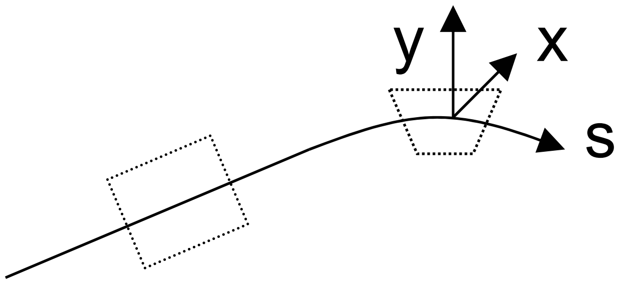




Coordinate System

Change dependent variable from time, *t*, to longitudinal position, *s*

Coordinate system used to describe the motion is usually locally Cartesian or cylindrical



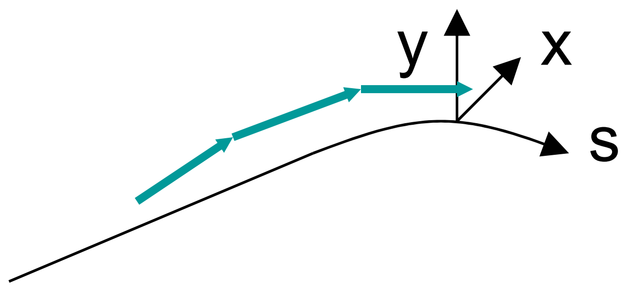
Typically the coordinate system chosen is the one that allows the easiest field representation

Integrate

Integrate through the elements

Use the following coordinates*

$$x,\quad x' = \frac{dx}{ds},\quad y,\quad y' = \frac{dy}{ds},\quad \delta = \frac{\Delta p}{p_0}, \tau = \frac{\Delta L}{L}$$



Note sometimes one uses canonical momentum rather than x' and y'

General equations of motion

- The equations of motion within an element is

$$\boxed{\begin{aligned} x'' &= \frac{1}{\rho}(1-\frac{x}{\rho}) - \frac{qB_y}{P} \\ y'' &= \frac{qB_x}{P} \end{aligned}}$$

- The fields have to be defined

Equations of motion – Linear fields

- The equations become

$$\boxed{\begin{aligned} x'' - \left(k(s) - \frac{1}{\rho(s)^2}\right)x &= \frac{1}{\rho(s)}\frac{\Delta P}{P} \\ y'' + k(s)\,y &= 0 \end{aligned}}$$

- Inhomogeneous equations with s-dependent coefficients
- Note that the term **$1/\rho^2$** corresponds to the dipole **weak focusing**
- The term **ΔP/(Pρ)** is present for off-momentum particles

Hill's equations

- Solutions are combination of the ones from the homogeneous and inhomogeneous equations
- Consider particles with the design momentum. The equations of motion become

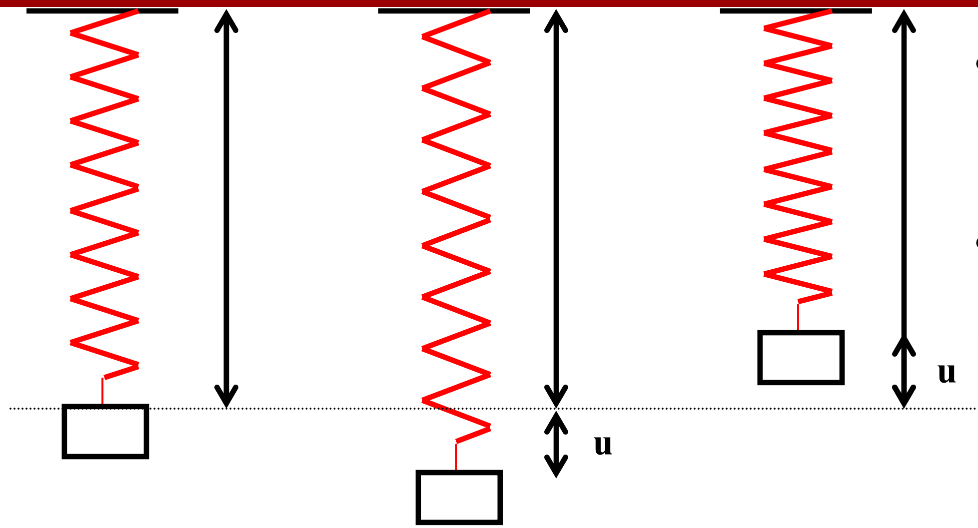
$$\boxed{\begin{aligned} x'' + K_x(s)\ x &= 0 \\ y'' + K_y(s)\ y &= 0 \end{aligned}}$$

with $$K_x(s) = -\left(k(s) - \frac{1}{\rho(s)^2}\right)\ ,\quad K_y(s) = k(s)$$

George Hill

- **Hill's equations of linear transverse particle motion**
- Linear equations with s-dependent coefficients (harmonic oscillator with time dependent frequency)
- In a ring or in transport line with symmetries, coefficients are periodic $K_x(s) = K_x(s+C)\ ,\ \ K_y(s) = K_y(s+C)$
- Not feasible to get analytical solutions for all accelerator

Harmonic oscillator – spring



- Consider K(s) = k_0 = constant

$$u'' + k_0\ u = 0$$

- Equations of harmonic oscillator with solution

$$\begin{aligned} u(s) &= C(s)\ u(0) + S(s)\ u'(0) \\ u'(s) &= C'(s)u(0) + S'(s)u'(0) \end{aligned}$$

with

$$C(s) = \cos(\sqrt{k_0}s)\ , \quad S(s) = \frac{1}{\sqrt{k_0}}\sin(\sqrt{k_0}s) \qquad \text{for } k_0 > 0$$

$$C(s) = \cosh(\sqrt{|k_0|}s)\ , \ S(s) = \frac{1}{\sqrt{|k_0|}}\sinh(\sqrt{|k_0|}s) \qquad \text{for } k_0 < 0$$

- Note that the solution can be written in **matrix** form

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \begin{pmatrix} u(0) \\ u'(0) \end{pmatrix}$$

Matrix formalism

- General **transfer matrix** from s_0 to s

$$\begin{pmatrix} u \\ u' \end{pmatrix}_s = \mathcal{M}(s|s_0)\begin{pmatrix} u \\ u' \end{pmatrix}_{s_0} = \begin{pmatrix} C(s|s_0) & S(s|s_0) \\ C'(s|s_0) & S'(s|s_0) \end{pmatrix}\begin{pmatrix} u \\ u' \end{pmatrix}_{s_0}$$

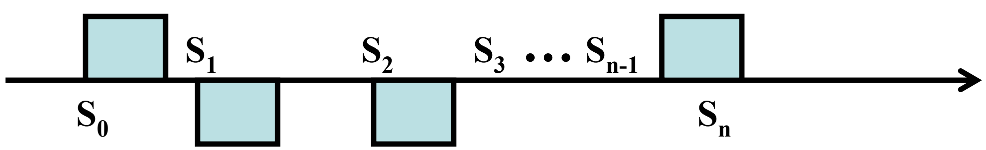
- Note that $\det(\mathcal{M}(s|s_0)) = C(s|s_0)S'(s|s_0) - S(s|s_0)C'(s|s_0) = 1$

  which is always true for conservative systems

- Note also that $\mathcal{M}(s_0|s_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathcal{I}$

- The accelerator can be build by a series of matrix multiplications

$$\mathcal{M}(s_n|s_0) = \mathcal{M}(s_n|s_{n-1}) \ldots \mathcal{M}(s_3|s_2) \cdot \mathcal{M}(s_2|s_1) \cdot \mathcal{M}(s_1|s0)$$



from $\mathbf{s_0}$ to $\mathbf{s_1}$

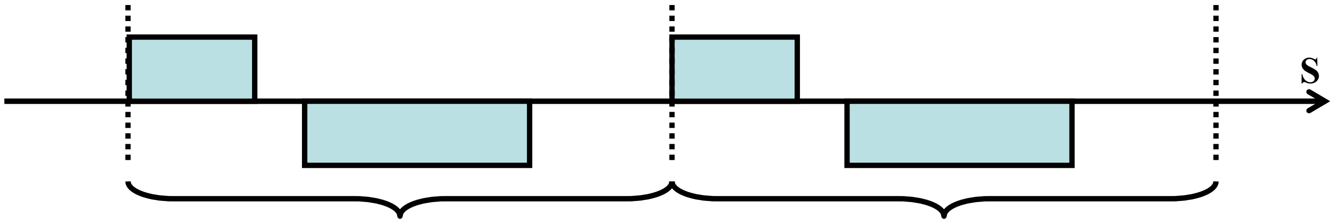
from $\mathbf{s_0}$ to $\mathbf{s_2}$

from $\mathbf{s_0}$ to $\mathbf{s_3}$

from $\mathbf{s_0}$ to $\mathbf{s_n}$

Symmetric lines

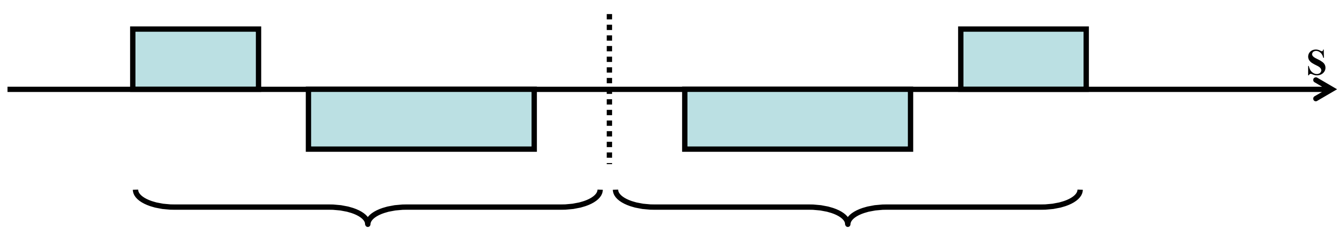
- System with normal symmetry



$$\mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\mathcal{M}_{\text{tot}} = \mathcal{M} \cdot \mathcal{M} = \mathcal{M}^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix}$$

- System with mirror symmetry



$$\mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \mathcal{M}_r = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$$

$$\mathcal{M}_{\text{tot}} = \mathcal{M}_r \cdot \mathcal{M} = \begin{pmatrix} ad + bc & 2bd \\ 2ac & ad + cb \end{pmatrix}$$

4x4 Matrices

- Combine the matrices for each plane

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} C_x(s) & S_x(s) \\ C'_x(s) & S'_x(s) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

$$\begin{pmatrix} y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} C_y(s) & S_y(s) \\ C'_y(s) & S'_y(s) \end{pmatrix} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}$$

to get a total 4x4 matrix

$$\begin{pmatrix} x(s) \\ x'(s) \\ y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} C_x(s) & S_x(s) & 0 & 0 \\ C'_x(s) & S'_x(s) & 0 & 0 \\ 0 & 0 & C_y(s) & S_y(s) \\ 0 & 0 & C'_y(s) & S'_y(s) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \end{pmatrix}$$

Transfer matrix of a drift

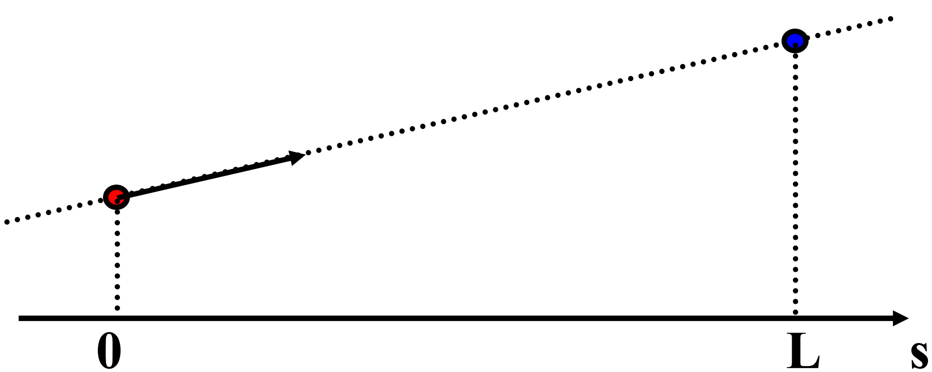
- Consider a drift (no magnetic elements) of length $L=s-s_0$

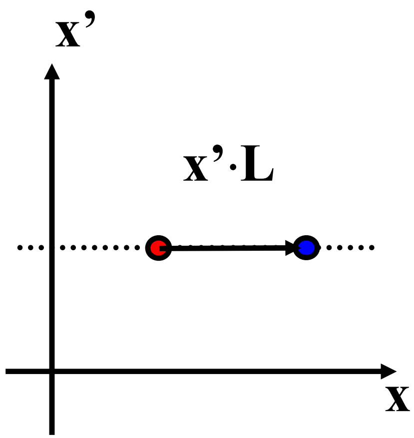
$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} 1 & s-s_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u'_0 \end{pmatrix}$$

$$\boxed{\mathcal{M}_{\mathrm{drift}}(s|s_0) = \begin{pmatrix} 1 & s-s_0 \\ 0 & 1 \end{pmatrix}}$$

$$\begin{aligned} u(s) &= u_0 + (s-s_0)u'_0 = u_0 + Lu'_0 \\ u'(s) &= u'_0 \end{aligned}$$

- Position changes if there is a slope. Slope remains unchanged





Focusing - defocusing thin lens

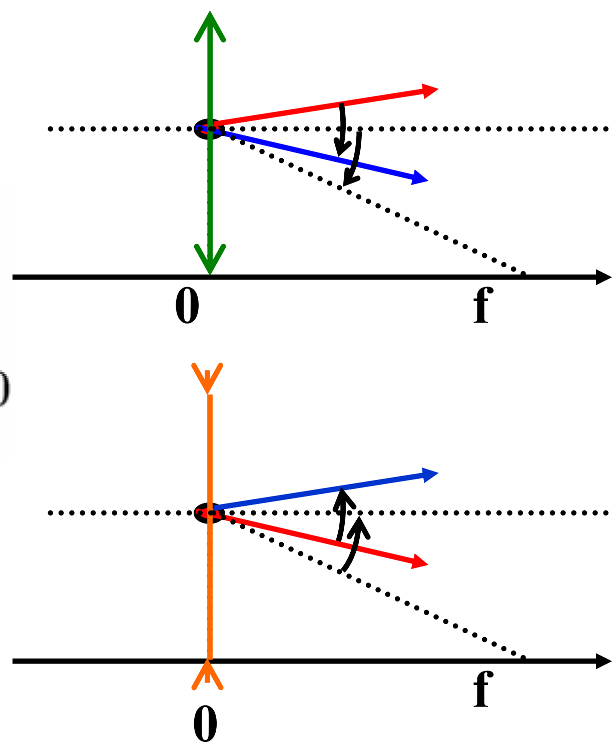
- Consider a lens with focal length ±**f**

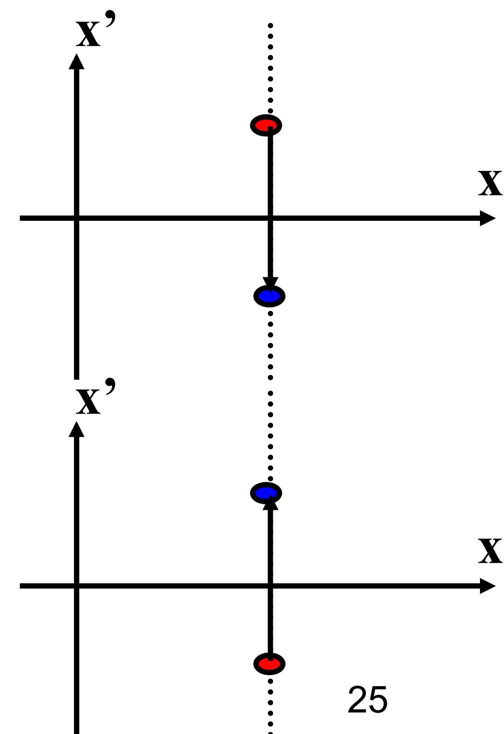
$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \mp\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u'_0 \end{pmatrix}$$

$$\boxed{\mathcal{M}_{\text{lens}}(s|s_0) = \begin{pmatrix} 1 & 0 \\ \mp\frac{1}{f} & 1 \end{pmatrix}}$$

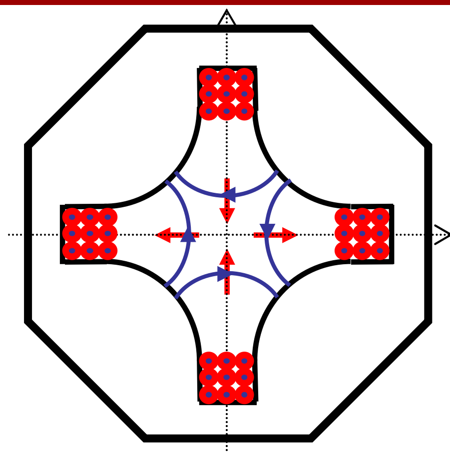
- Slope diminishes (focusing) or increases (defocusing). Position remains unchanged

$$\begin{aligned} u(s) &= u_0 \\ u'(s) &= u_0 \mp \frac{1}{f} u'_0 \end{aligned}$$





Quadrupole



- Consider a quadrupole magnet of length **L**. The field is

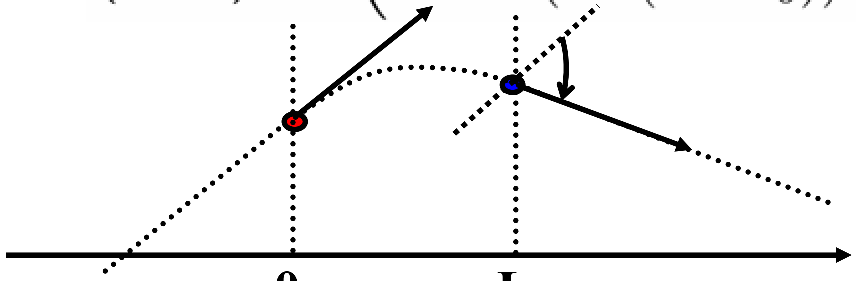
$$B_y = -G(s)x\ ,\quad B_x = -G(s)y$$

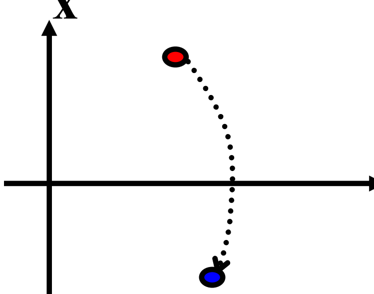
- with normalized quadrupole gradient (in **m^{-2}**)

$$k = \frac{G}{B_0\rho}$$

The transport through a quadrupole is

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{k}(s-s_0)) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}(s-s_0)) \\ \sqrt{k}\sin(\sqrt{k}(s-s_0)) & \cos(\sqrt{k}(s-s_0)) \end{pmatrix} \begin{pmatrix} u_0 \\ u_0' \end{pmatrix}$$





Sector Dipole

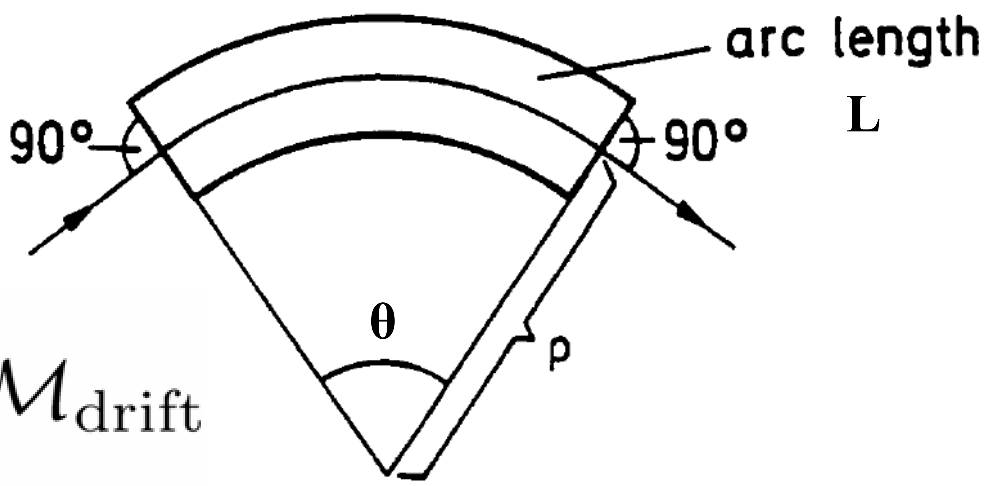
- Consider a dipole of length **L**. By setting in the focusing quadrupole matrix

$$k = \frac{1}{\rho^2} > 0$$

the transfer matrix for a sector dipole becomes

$$\boxed{\mathcal{M}_{\text{sector}} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{pmatrix}}$$

with a bending radius $\theta = \frac{L}{\rho}$

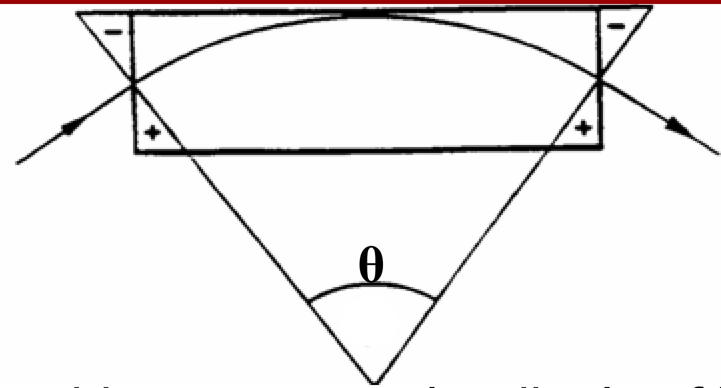


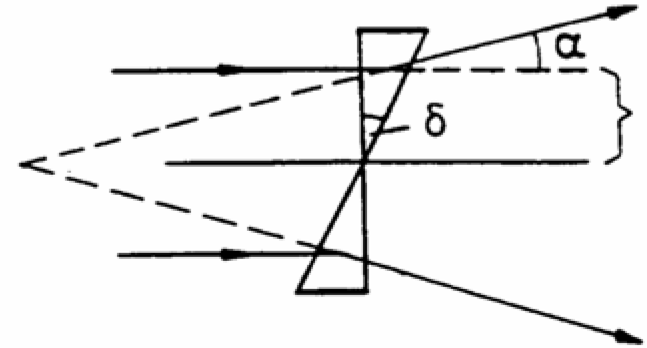
- In the non-deflecting plane $\frac{1}{\rho} \to 0$

$$\mathcal{M}_{\text{sector}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} = \mathcal{M}_{\text{drift}}$$

- This is a **hard-edge** model. In fact, there is some **edge focusing** in the vertical plane

Rectangular Dipole





- Consider a rectangular dipole of length **L**. At each edge, the deflecting angle is

$$\alpha = \frac{\Delta L}{\rho} = \frac{\theta \tan\delta}{\rho}$$

$$\frac{1}{f} = \frac{\tan\delta}{\rho}$$

It acts as a thin defocusing lens with focal length

- The transfer matrix is

$$\mathcal{M}_{\text{rect}} = \mathcal{M}_{\text{edge}} \cdot \mathcal{M}_{\text{sector}} \cdot \mathcal{M}_{\text{edge}}$$

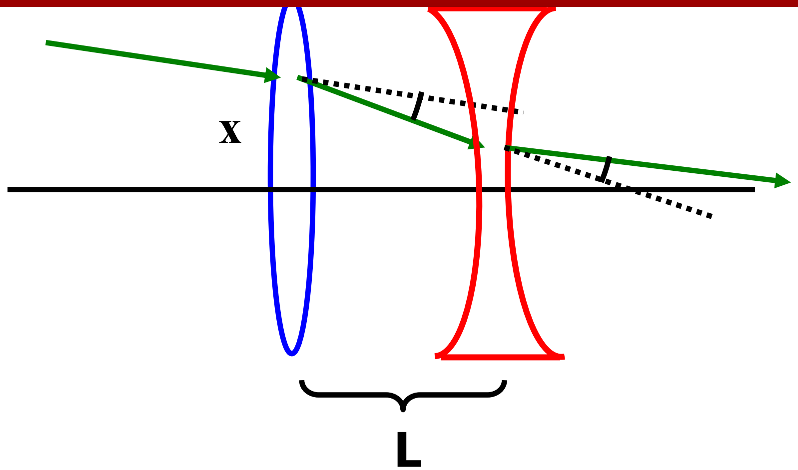
with

$$\mathcal{M}_{\text{edge}} = \begin{pmatrix} 1 & 0 \\ \frac{\tan(\delta)}{\rho} & 1 \end{pmatrix}$$

- For **θ<<1**, **δ=θ/2**.
- In deflecting plane (like **drift**) in non-deflecting plane (like **sector**)

$$\mathcal{M}_{x;\text{rect}} = \begin{pmatrix} 1 & \rho\sin\theta \\ 0 & 1 \end{pmatrix} \quad \mathcal{M}_{y;\text{rect}} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{pmatrix}$$

Quadrupole Doublet and AG Focusing



- Consider a quadrupole doublet, i.e. two quadrupoles with focal lengths $\mathbf{f_1}$ and $\mathbf{f_2}$ separated by a distance **L**.
- In thin lens approximation the transport matrix is

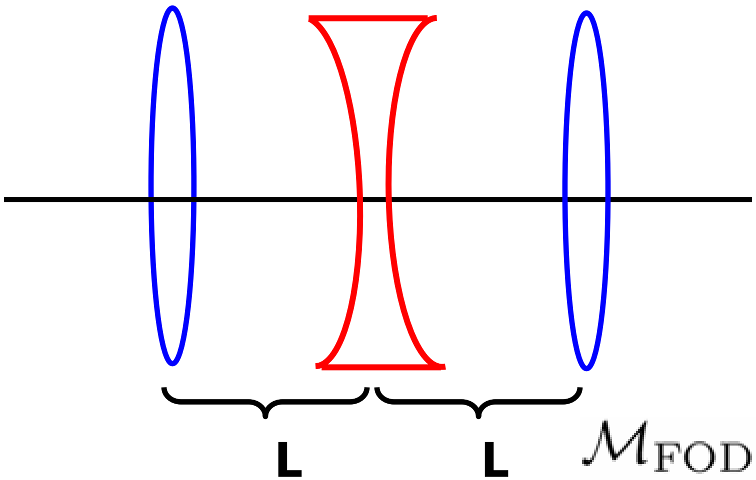
$$\mathcal{M}_{\text{doublet}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{f_1} & L \\ -\frac{1}{f^\star} & 1 - \frac{L}{f_2} \end{pmatrix}$$

with the **total focal length**

$$\frac{1}{f^\star} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{L}{f_1\ f_2}$$

- Setting $\mathbf{f_1 = -f_2 = f}$ $\qquad \frac{1}{f^\star} = \frac{L}{f^2}$
- Alternating gradient focusing seems overall focusing
- This is only valid in thin lens approximation!!!

FODO Cell



- Consider a defocusing quadrupole "sandwiched" by two focusing quadrupoles with focal lengths **f**.
- The symmetric transfer matrix from center to center of focusing quads

$$\mathcal{M}_{\text{FODO}} = \mathcal{M}_{\text{HQF}} \cdot \mathcal{M}_{\text{drift}} \cdot \mathcal{M}_{\text{QD}} \cdot \mathcal{M}_{\text{drift}} \cdot \mathcal{M}_{\text{HQF}}$$

with the transfer matrices

$$\mathcal{M}_{\text{HQF}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}, \quad \mathcal{M}_{\text{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}, \quad \mathcal{M}_{\text{QD}} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

- The total transfer matrix is

$$\boxed{\mathcal{M}_{\text{FODO}} = \begin{pmatrix} 1 - \frac{L^2}{2f^2} & 2L(1 + \frac{L}{2f}) \\ \frac{L}{2f^2}(1 - \frac{L}{2f}) & 1 - \frac{L^2}{2f^2} \end{pmatrix}}$$

Magnetic Multipole Expansion

- From Gauss law of magnetostatics, we construct a vector potential

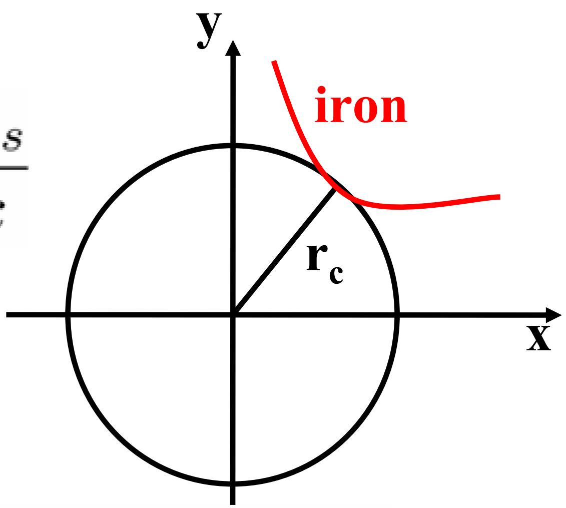
$$\nabla \cdot \mathbf{B} = 0 \quad \rightarrow \quad \exists \mathbf{A} : \quad \mathbf{B} = \nabla \times \mathbf{A}$$

- Assuming a 2D field in **x** and **y**, the vector potential has only one component $\mathbf{A}_s$
- The Ampere's law in vacuum (inside the beam pipe)

$$\nabla \times \mathbf{B} = 0 \quad \rightarrow \quad \exists V : \quad \mathbf{B} = -\nabla V$$

- Using the previous equations one finds the conditions which are Riemann conditions of an analytic function.

$$B_x = -\frac{\partial V}{\partial x} = \frac{\partial A_s}{\partial y} , \qquad B_y = -\frac{\partial V}{\partial y} = -\frac{\partial A_s}{\partial x}$$



- There exist a complex potential of **z = x+iy** with a power series expansion convergent in a circle with radius $|z| = r_c$ (distance from iron yoke)

$$\mathcal{A}(x+iy) = A_s(x,y) + iV(x,y) = \sum_{n=1}^{\infty} \kappa_n z^n = \sum_{n=1}^{\infty} (\lambda_n + i\mu_n)(x+iy)^n$$

Magnetic Multipole Expansion

- From the complex potential we can derive the fields

$$B_y + iB_x = -\frac{\partial}{\partial x}(A_s(x,y) + iV(x,y)) = -\sum_{n=1}^{\infty} n(\lambda_n + i\mu_n)(x+iy)^{n-1}$$

- Setting $b_n = -n\lambda_n$, $a_n = n\mu_n$ we have

$$B_y + iB_x = \sum_{n=1}^{\infty}(b_n - ia_n)(x+iy)^{n-1}$$

- Define normalized units $b'_n = \frac{b_n}{10^{-4}B_0} r_0^{n-1}$, $a_n = \frac{a_n}{10^{-4}B_0} r_0^{n-1}$

on a reference radius, 10^{-4} of the main field to get

$$B_y + iB_x = 10^{-4}B_0 \sum_{n=1}^{\infty}(b'_n - ia'_n)\left(\frac{x+iy}{r_0}\right)^{n-1}$$

- **Note**: **n'=n-1** the American convention

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L3 Possible Homework

• Derive the thin matrix representation for a focusing quadrupole starting from the "thick" element matrix. Hint: calculate the limit for the matrix when the quadrupole length approaches zero while the integrated magnetic field is kept constant.

• Suppose that a particle traverses, first, a thin focusing lens with a focal length F; second, a drift of length L; third, a thin defocusing lens with focal length F; and, fourth, another drift of length L. Calculate the matrix for this cell.

• Consider a system made up of two thin lenses each of focal length F, one focusing and one defocusing, separated by a distance L. Show that the system is focusing if $|F|>L$.